ON THE DIOPHANTINE EQUATION $1^k + 2^k + \cdots + x^k = y^n$

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Abstract. In this paper, we resolve a conjecture of Schäffer on the solvability of Diophantine equations of the shape

$$1^k + 2^k + \cdots + x^k = y^n,$$

for $1 \leq k \leq 11$. Our method, which may, with a modicum of effort, be extended to higher values of k, combines a wide variety of techniques, classical and modern, in Diophantine analysis.

1. Introduction

A classical question of Lucas [19] is whether the Diophantine equation

$$1^2 + 2^2 + \cdots + x^2 = y^2$$

has solutions in positive integers other than $(x, y) = (1, 1)$ and $(x, y) = (24, 70)$. Subsequent attempts by Lucas [20] and Moret-Blanc [24] to resolve this suffer from various defects and it was not until a number of years later that Watson [33] was able to correctly answer the question, in the negative. For more recent proofs, from a variety of perspectives, the reader is directed to [1], [8], [17] and [21].

In 1956, Schäffer [28] considered the more general equation

$$S_k(x) = y^n \tag{1}$$

where, here and subsequently, we write

$$S_k(x) = 1^k + 2^k + \cdots + x^k$$

for k a positive integer. He showed, for fixed $k \geq 1$ and $n \geq 2$, that (1) possesses at most finitely many solutions in positive integers x and y,

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unless

$$(k,n) \in \{(1,2),(3,2),(3,4),(5,2)\}, \tag{2}$$

where, in each case, there are infinitely many such solutions. In essence, this amounts to showing that equation (1) defines, for pairs (k,n) not in (2), a curve of positive genus, whereby the statement is a consequence of Siegel's theorem on integral points on curves. Since Siegel's work is ineffective, the same is true of Schäffer's proof; i.e. for given (k,n) not in (2), it is not *a priori* possible to determine the (finite) set of solutions to (1). Since $(x,y) = (1,1)$ satisfies (1) for each k and n, we will refer to this solution as *trivial*. Subsequently, Győry, Tijdeman and Voorhoeve [12] provided an effective proof that for fixed $k \geq 2$ with $k \notin \{3,5\}$, (1) has only finitely many non-trivial solutions in positive integers x, y and n with $n \geq 2$. Further, Pintér [25] showed that, in case of non-trivial solutions, we have $n < ck^2 \log(2k)$ with an effectively computable absolute constant $c > 0$. These last two results, while effective, rely, as is often the case in such situations, upon lower bounds for linear forms in logarithms of algebraic numbers which lead to implicit constants of such a size as to make explicit solution of (1) impractical.

For certain values of $k \leq 11$ and $n \geq 2$, howevere, Schäffer [28] was able to show that equation (1) has only the trivial solution. In particular, he obtained such a result for $k \in \{1,5\}$ and $n = 4$, for $k = 3$ and $n = 8$, for $k \in \{4,6,8,9,10\}$ and $n = 2$, for $k \leq 11$ and $n \in \{3,5\}$, and for $k \leq 11$ with $k \neq 10$ and certain regular primes n. Further, Schäffer [28] conjectured that for $k \geq 1$ and $n \geq 2$ with (k,n) not in the set (2), (1) has only one non-trivial solution, namely, $(k,n,x,y) = (2,2,24,70)$. Recently, Jacobson, Pintér and Walsh [14] verified Schäffer's conjecture (cf. Section 3) for $n = 2$ and even values of k with $k \leq 58$. For further results on this topic, including a variety of generalizations, we refer the reader to [6], [10], [12], [30], [31], [32] and to the notes at the end of Chapter 10 of [29].

The purpose of this paper is to prove Schäffer's conjecture completely for $k \leq 11$ (which includes all the values considered by Schäffer) and, most importantly, for arbitrary n. More precisely, we demonstrate the following

Theorem 1.1. *For $1 \leq k \leq 11$ and (k,n) not in the set (2), equation (1) has only the trivial solution, unless $k = 2$, in which case there is the additional solution $(n,x,y) = (2,24,70)$.*

The main interest in this result is that it affords us an opportunity to employ a combination of virtually every technique in modern

Diophantine analysis, including local methods, a classical reciprocity theorem in cyclotomic fields, lower bounds for linear forms in logarithms of algebraic numbers, the hypergeometric method of Thue and Siegel and results on ternary equations based upon Galois representations and modular forms. It is a rare and perhaps rather special situation where one can explicitly solve superelliptic equations of as high degree as we encounter here. We accomplish this through application of a new method for solving certain high degree Thue equations, itself a notoriously difficult problem.

In the sections that follow, we begin with some basic facts about the polynomials $S_k(x)$, and then proceed with the proof of Theorem 1.1 in the rather different, as it transpires, cases when n is or is not a power of 2.

2. The polynomials $S_k(x)$

The polynomials $S_k(x)$ are intimately connected to Bernoulli numbers. Let us begin by stating some of their well-known properties, of which we will have later need; see e.g. [26] for details. If $k = 1$, then $S_1(x) = x(x+1)/2$, while, if $k > 1$, we can write

$$S_k(x) = \begin{cases} \frac{1}{C_k}\, x^2(x+1)^2 T_k(x) & \text{if } k > 1 \text{ is odd} \\ \\ \frac{1}{C_k}\, x(x+1)(2x+1) T_k(x) & \text{if } k > 1 \text{ is even} \end{cases}$$

where C_k is a positive integer and $T_k(x)$ is a polynomial with integer coefficients. For $2 \le k \le 11$, we explicitly compute $S_k(x)$ to find that C_k and $T_k(x)$ are as follows :

k	C_k	$T_k(x)$
2	6	1
3	4	1
4	30	$3x^2 + 3x - 1$
5	12	$2x^2 + 2x - 1$
6	42	$3x^4 + 6x^3 - 3x + 1$
7	24	$3x^4 + 6x^3 - x^2 - 4x + 2$
8	90	$5x^6 + 15x^5 + 5x^4 - 15x^3 - x^2 + 9x - 3$
9	20	$(x^2 + x - 1)(2x^4 + 4x^3 - x^2 - 3x + 3)$
10	66	$(x^2 + x - 1)(3x^6 + 9x^5 + 2x^4 - 11x^3 + 3x^2 + 10x - 5)$
11	24	$2x^8 + 8x^7 + 4x^6 - 16x^5 - 5x^4 + 26x^3 - 3x^2 - 20x + 10$

Notice that $C_k \equiv 0 \pmod{k+1}$ in each case. Further, an elementary calculation reveals that for every positive integer $x > 1$

$$\gcd(x(x+1), T_k(x)) = 1 \quad \text{if} \quad k \in \{2, 3, 4, 5, 6\}$$

while, for larger values of k, $\gcd(x(x+1), T_k(x))$ divides

$$\begin{cases} 2 & \text{if } k = 7 \\ 3 & \text{if } k \in \{8,9\} \\ 5 & \text{if } k = 10 \\ 10 & \text{if } k = 11. \end{cases}$$

Further, if p is a prime with $p | \gcd(x(x+1), T_k(x))$, then $p \| T_k(x)$. Similarly, it is easy to show that $\gcd(C_k, T_k(x))$ divides

$$\begin{cases} 1 & \text{if } k \in \{2,3\} \\ 3 & \text{if } k = 5 \\ 5 & \text{if } k \in \{4,9\} \\ 6 & \text{if } k \in \{7,11\} \\ 7 & \text{if } k = 6 \\ 11 & \text{if } k = 10 \\ 15 & \text{if } k = 8. \end{cases}$$

Writing $\mathrm{ord}_p(m)$ for the largest integer k such that p^k divides m, we have that $\mathrm{ord}_3(T_k(x)) \le 1$ for $2 \le k \le 11$, while $\mathrm{ord}_5(T_9(x)) \le 1$ and

$$\mathrm{ord}_2(T_7(x)) = \mathrm{ord}_2(T_{11}(x)) = \mathrm{ord}_3(T_8(x)) = 1.$$

Finally, we have that $\gcd(2x+1, T_k(x))$ divides

$$\begin{cases} 1 & \text{if } k = 2 \\ 7 & \text{if } k = 4 \\ 31 & \text{if } k = 6 \\ 3 \cdot 127 = 381 & \text{if } k = 8 \\ 5 \cdot 7 \cdot 73 = 2555 & \text{if } k = 10. \end{cases}$$

Again, if p is a prime dividing $\gcd(2x+1, T_k(x))$, then $p \| T_k(x)$.

In the proof of Theorem 1.1 we shall distinguish two cases. First we deal with the situation when n is a power of 2, by explicitly solving two Diophantine equations of the form

$$T_k(x) = Cz^2$$

for certain fixed values of C. For the values of k under consideration, each of these is equivalent to determining the "integer points" on a particular model of an elliptic curve, a problem that is, nowadays, frequently routine (as, indeed, is the case for us). In Section 4, we suppose that there is an odd prime factor of n and concern ourselves with equations of the form

$$x(x+1) = Cz^n,$$

again for certain choices of C, depending upon the various properties of $S_k(x)$ outlined in this section.

3. Theorem 1.1 for n a power of 2

For n a power of 2 and for $k \le 10$ with $k \neq 7$, Theorem 1.1 is an immediate consequence of the following result of Watson [33] (in case $k = 2$) and Schäffer [28] (otherwise) :

Lemma 3.1. *For $k \in \{2, 4, 6, 8, 9, 10\}$ and $n = 2$, the only non-trivial solution (x, y) to equation (1) is given by $(x, y, k) = (24, 70, 2)$. Further, for $(k, n) \in \{(1, 4), (5, 4), (3, 8)\}$, equation (1) has no non-trivial solution.*

It suffices, then, to deal with the cases when $k \in \{7, 11\}$. It follows in both cases that

$$T_k(x) = \frac{1}{4} C_k u^2 \tag{3}$$

for some $u \in \mathbb{N}$. If $k = 7$, we thus have

$$3x^4 + 6x^3 - x^2 - 4x + 2 = 6u^2. \tag{4}$$

Writing $v = x(x + 1)$, we conclude that

$$T_{11}(x) = 2v^4 - 8v^3 + 17v^2 - 20v + 10$$

whereby

$$2v^4 - 8v^3 + 17v^2 - 20v + 10 = 6u^2. \tag{5}$$

These correspond to models for elliptic curves of conductor 9792 and 135360, respectively. Finding integer solutions to elliptic equations (4) and (5) is nowadays a relatively straightforward matter via, say, lower bounds for linear forms in elliptic logarithms. Such a method (it is imprecise, in this situation, to use the word "algorithm"), as implemented in the computational package MAGMA [22], asserts that the only integral solutions to these equations are given by $(x, u) = (-2, \pm 1), (1, \pm 1)$ and $(v, u) = (2, \pm 1)$, respectively. None of these correspond to non-trivial solutions for our original problem. This completes the proof of Theorem 1.1 in case n is a power of 2. We note here that extending this result to larger odd values of k (i.e. with $k \ge 13$) requires the determination of integral points on hyperelliptic curves of genus as large as $\frac{k-5}{4}$. This, while not a routine matter, may often be successfully carried out via effective Coleman-Chabauty methods.

4. Theorem 1.1 for n with an odd prime factor : an upper bound for n

We will now suppose that $n \ge 3$ is prime. In view of Schäffer's result [28] it suffices to deal with the case when $n \ge 7$. The results of Section

2 imply that

$$x(x+1) = 2 \cdot 3^{\delta_3} \cdot 5^{\delta_5} \cdot 7^{\delta_7} \cdot 11^{\delta_{11}} \cdot y_1^n \tag{6}$$

for some positive integer y_1, where the choices for the δ_i's are as follows:

k	δ_3	δ_5	δ_7	δ_{11}
$1,3$	0	0	0	0
2	$0,1$	0	0	0
$4,8$	$0,1$	$0,1$	0	0
$5,7$	$0,\frac{n+1}{2}$	0	0	0
6	$0,1$	0	$0,1$	0
9	$0,\frac{n-1}{2}$	$0,\frac{n+1}{2}$	0	0
10	$0,1$	$0,n-1$	0	$0,1$
11	$0,\frac{n+1}{2}$	$0,\frac{n-1}{2}$	0	0

A priori, for each n, this leads to 18 possible equations of the form (6). We may reduce this number to 14 by appealing to a recent result of the first author [3]:

Lemma 4.1. *If $m, y \geq 1$, $n \geq 3$ and $\alpha, \beta, t \geq 0$ are integers for which*

$$m(m+2^t) = 2^\alpha \cdot 3^\beta \cdot y^n,$$

then

$$m \in \{2^t, 2^{t\pm 1}, 3 \cdot 2^t, 2^{t\pm 3}\}.$$

Taking $t = 0$ and $m = x$, we may thus assume that $\max\{\delta_5, \delta_7, \delta_{11}\} > 0$, unless $x \in \{2, 3, 4, 9\}$. For these values of x, it is a routine matter to verify that $S_k(x)$ is not an nth power for $n > 5$ odd and $1 \leq k \leq 11$. It follows that we may suppose, here and henceforth, that $k \in \{4, 6, 8, 9, 10, 11\}$.

To estimate from above the unknown exponent n in the remaining equations (6), we turn to a lower bound for linear forms in logarithms of two algebraic numbers (Theorem 2 of Laurent, Mignotte and Nesterenko [16]) :

Lemma 4.2. *Let α_1 and α_2 be two positive real algebraic numbers. Consider*

$$\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1,$$

where b_1 and b_2 are positive rational integers. Put $D = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]$ and suppose that $\log \alpha_1$ and $\log \alpha_2$ are linearly independent over $\mathbb{Q}$. For any $\rho > 1$, take

$$h \geq \max\left\{\frac{D}{2}, 5\lambda, D\left(\log\left(\frac{b_1}{a_2} + \frac{b_2}{a_1}\right) + \log\lambda + 1.56\right)\right\},$$

$$a_i \geq (\rho - 1)|\log \alpha_i| + 2Dh(\alpha_i), (i = 1, 2),$$

and

$$a_1 + a_2 \geq 4 \max\{1, \lambda\},$$
$$\frac{1}{a_1} + \frac{1}{a_2} \leq \min\{1, \lambda^{-1}\},$$

where $\lambda = \log \rho$. *Then*

$$\log|\Lambda| \geq \begin{array}{l} -\frac{a_1 a_2}{9\lambda} R^2 - \frac{2}{3}(a_1 + a_2)R - \frac{16}{3}\sqrt{2a_1 a_2} S^{3/2} \\ -\log\left(\frac{a_1 a_2}{\lambda} S^2\right) - \frac{3}{2}\lambda - 2h - \frac{3}{20} \end{array}$$

with

$$R = \frac{4h}{\lambda} + 4 + \frac{\lambda}{h} \quad \textit{and} \quad S = 1 + \frac{h}{\lambda}.$$

Here, if α is an algebraic number of degree d over $\mathbb{Q}$, with minimal polynomial over $\mathbb{Z}$ given by

$$a \prod_{i=1}^{d} (X - \alpha^{(i)}),$$

where $\alpha^{(i)} \in \mathbb{C}$, we write

$$h(\alpha) = \frac{1}{d}\left(\log|a| + \sum_{i=1}^{d} \log \max(1, |\alpha^{(i)}|)\right)$$

for the absolute logarithmic height of α. The above lemma enables us to prove the following

Proposition 4.3. *Equation (1) has no solutions in positive integers* $(x, y) \neq (1, 1)$ *with* $1 \leq k \leq 11$ *and* $n > 4000$ *prime.*

It is convenient at this stage to introduce a result that will be used throughout the remainder of the paper; this is Theorem 1.1 of Bennett [2].

Lemma 4.4. *If* A, B *and* n *are integers with* $AB \neq 0$ *and* $n \geq 3$, *then the equation*

$$AX^n - BY^n = \pm 1$$

has at most one solution in positive integers (X, Y). *In particular, for* $A \geq 1$ *and* $B = A + 1$, *the above equation has precisely the solution* $(X, Y) = (1, 1)$ *in positive integers.*

This result is based, primarily, on the hypergeometric method of Thue and Siegel; i.e. on Padé approximation to powers of $(1 - z)^{1/n}$.

Proof of Proposition 4.3. From (6), if there is a nontrivial solution to (1) with $1 \leq k \leq 11$ and $n > 4000$ an odd prime, there necessarily exist integers a and b with $|ab| > 1$ and

$$Aa^n - Bb^n = 1, \tag{7}$$

where

$$AB = 2 \cdot 3^{\delta_3} \cdot 5^{\delta_5} \cdot 7^{\delta_7} \cdot 11^{\delta_{11}}$$

and, without loss of generality, $A < B$. It follows that AB is in the following list :

$$\begin{aligned} &10, 14, 22, 30, 42, 66, 2 \cdot 5^{n-1}, 6 \cdot 5^{n-1}, 22 \cdot 5^{n-1}, \\ &66 \cdot 5^{n-1}, 2 \cdot 5^{\frac{n\pm1}{2}}, 2 \cdot 3^{\frac{n\mp1}{2}} \cdot 5^{\frac{n\pm1}{2}}. \end{aligned} \tag{8}$$

These choices lead to 50 possibilities for A, B in (7) (or 48 if we discount the pairs $(A, B) = (5, 6)$ and $(6, 7)$ which may be treated via Lemma 4.4). We will provide full details for 3 of these 48 equations, including the case which leads to our weakest bound upon n; these examples include all the features of the general situation.

In terms of application of Lemma 4.2, the equations corresponding to (8) fall into three categories, depending on whether

$$AB \in \{10, 14, 22, 30, 42, 66\}, \tag{9}$$

$$AB \in \left\{2 \cdot 5^{n-1}, 6 \cdot 5^{n-1}, 22 \cdot 5^{n-1}, 66 \cdot 5^{n-1}\right\} \tag{10}$$

or

$$AB \in \left\{2 \cdot 5^{\frac{n\pm1}{2}}, 2 \cdot 3^{\frac{n\mp1}{2}} \cdot 5^{\frac{n\pm1}{2}}\right\}. \tag{11}$$

We will detail our argument for the pair (A, B) that leads to the weakest upper bound upon n, in each of these three cases. Before we proceed, we note that to utilize the full strength of Lemma 4.2, it will prove helpful to have a decent lower bound upon $|a|$. For example, if we suppose that $a^n - Db^n = 1$ with n coprime to $\phi(D)$ and $|a| > 1$, then, if $p|D$, $a^n \equiv 1 \pmod{p}$ and so $a \equiv 1 \pmod{p}$. It follows that

$$a \equiv 1 \ (\bmod \prod_{p|D} p)$$

and so (since we assume $|a| > 1$),

$$|a| \geq \prod_{p|D} p - 1.$$

If $A, B > 1$, we can, in any case, via a simple algorithm, produce lower bounds upon $\min\{|a|, |b|\}$. For example, if

$$2 \cdot 3^{(n+1)/2} a^n - 5^{(n-1)/2} b^n = 1, \tag{12}$$

where we may suppose that a and b are nonzero integers, then

$$\left| \sqrt{60} \left(\frac{\sqrt{3}a}{\sqrt{5}b} \right)^n - 1 \right| = \frac{1}{5^{(n-1)/2} |b|^n}$$

and so, from the inequality $|x| < 2\,|e^x - 1|$, valid for real x,

$$\left|\log 60 - n \log\left(\frac{5b^2}{3a^2}\right)\right| < \frac{4\sqrt{5}}{(5b^2)^{n/2}}. \tag{13}$$

Dividing by n and assuming $n > 4000$, this implies that

$$0 < \log\left(\frac{5b^2}{3a^2}\right) < 0.002.$$

It is easy to show that the smallest $|a|$ (with a coprime to 10) for which such an inequality is satisfied is $|a| = 71$ (whence $|b| = 55$). We may thus suppose, for this example, that $5b^2 \geq 15125$. Similar arguments may be applied to our other equations.

Let us now consider AB as in (9). For these pairs, the largest upper bound upon n corresponds to the choice $A = 1, B = 66$. In this situation, we have

$$a^n - 66b^n = 1$$

for a and b integers with $ab > 0$. Then

$$|\log 66 - n\log(a/b)| < 2 \cdot |1 - 66\,(b/a)^n| < 2/|a|^n. \tag{14}$$

Assuming $n > 4000$ and $|a| \geq 65$, we choose, in the notation of Lemma 4.2,

$$\rho = 12,\ \lambda = \log 12,\ \ b_1 = 1,\ b_2 = n,\ \alpha_1 = 66,\ \alpha_2 = a/b,$$

$$a_1 = 54.47,\ a_2 = 2.01 \log|a| \ \text{ and } \ h = \log n.$$

Then

$$R < 2.13 \log n \ \text{ and } \ S < 0.53 \log n,$$

whereby

$$\frac{a_1 a_2}{9\lambda} R^2 < 22.22 \log|a| \log^2 n,$$

$$\frac{2}{3}(a_1 + a_2)R < (77.35 + 2.86 \log|a|) \log n < 2.58 \log|a| \log^2 n,$$

$$\frac{16}{3}\sqrt{2a_1a_2}S^{3/2} < 30.46\,(\log|a|)^{1/2}\,(\log n)^{3/2} < 5.18 \log|a| \log^2 n$$

and

$$\log\left(\frac{a_1a_2}{\lambda}S^2\right) + \frac{3}{2}\lambda + 2h + \frac{3}{20} < 0.10 \log|a| \log^2 n.$$

We may thus apply Lemma 4.2 to conclude that

$$|\log 66 - n\log(a/b)| > \exp\left(-30.08 \log|a| \log^2 n\right). \tag{15}$$

Combining (14) and (15) implies that

$$n \log|a| < 30.08 \log|a| \log^2 n + \log 2$$

and so
$$\frac{n}{\log^2 n} < 30.09,$$
whence $n < 1653$, a contradiction.

The largest bound for the Thue equations corresponding to (10) comes from the pair $(A, B) = (66, 5^{n-1})$. Here, we have
$$66a^n - 5^{n-1}b^n = 1, \quad \text{with } \ ab > 0,$$
whence
$$|330(a/5b)^n - 1| = \frac{1}{5^{n-1}|b|^n}$$
and so
$$|\log 330 - n \log{(5b/a)}| < \frac{10}{(5|b|)^n}. \tag{16}$$
We will again assume that $n > 4000$ and, since $b \neq 0$, that $5|b| \geq 5$. Choose $\rho = 12$ (so that $\lambda = \log 12$) and
$$b_1 = 1, \; b_2 = n, \; \alpha_1 = 330, \; \alpha_2 = 5b/a,$$
$$a_1 = 75.39, \; a_2 = 2.01 \log 5|b| \; \text{ and } \; h = \log n.$$
Arguing as in the preceding example leads to the conclusion that $n < 3204$, again a contradiction.

As our final example, with AB as in (11), let us consider the equation (12) where we again assume $ab > 0$. This situation leads to the worst bound upon n of all the pairs A, B in (8). We may again suppose that $n > 4000$ and, as noted earlier in this section, that $5b^2 \geq 15125$. We will apply Lemma 4.2 to the linear form given in (13), choosing
$$\rho = 12, \; \lambda = \log 12, \; \; b_1 = 1, \; b_2 = n, \; \alpha_1 = 60, \; \alpha_2 = 5b^2/3a^2,$$
$$a_1 = 53.23, \; a_2 = 2.01 \log(5b^2) \; \text{ and } \; h = \log n.$$
We have, again,
$$R < 2.13 \log n \; \text{ and } \; S < 0.53 \log n,$$
whereby
$$\frac{a_1 a_2}{9\lambda} R^2 < 21.71 \log(5b^2) \log^2 n,$$
$$\frac{2}{3}(a_1 + a_2) R < \big(75.59 + 2.86 \log(5b^2)\big) \log n < 1.3 \log(5b^2) \log^2 n,$$
$$\frac{16}{3} \sqrt{2a_1 a_2} S^{3/2} < 30.11 \left(\log(5b^2)\right)^{1/2} (\log n)^{3/2} < 3.38 \log(5b^2) \log^2 n$$
and
$$\log\left(\frac{a_1 a_2}{\lambda} S^2\right) + \frac{3}{2}\lambda + 2h + \frac{3}{20} < 0.05 \log(5b^2) \log^2 n.$$

We may thus apply Lemma 4.2 to conclude that

$$\left| \log 60 - n \log \left(\frac{5b^2}{3a^2} \right) \right| > \exp\left(-26.44 \log(5b^2) \log^2 n \right). \tag{17}$$

Combining (13) and (17) implies that

$$\frac{n}{\log^2 n} < \frac{\log 80}{\log(5b^2)\log^2 n} + 52.88 < 52.89$$

and so $n < 3530$, a contradiction.

For the remaining pairs (A, B), we obtain stronger bounds for n than in this last case. In each situation, we may take $\rho = 12$. Details are available from the authors upon request. In fact, working carefully via case by case analysis to increase our lower bounds upon $|a|$, we may obtain a rather better bound than $n < 4000$ in all cases (and much better, in many – such a sharpening is relatively unimportant for our purposes). This completes the proof of Proposition 4.3.

It remains to treat the $48 \cdot 547 = 26256$ triples (A, B, n) with AB as in (8), $B > A+1$, and $7 \leq n < 4000$ prime. For "small" values of n, this is readily accomplished via known computational techniques (indeed, for a number of triples (A, B, n) with $7 \leq n \leq 19$, we will utilize this approach). For values of n greater than 100 or so, however,this is well out of range of current methods based on lower bounds for linear forms in logarithms and lattice basis reduction. In the next two sections, we will illustrate a new technique for handling such equations, based upon classical work on Fermat-type equations and the theory of Frey curves and modular forms (together with elementary arguments).

5. Local methods

As it transpires, the task of solving the remaining Thue equations of the shape (7), for A, B with $\min\{A, B\} > 1$ is relatively routine. These correspond to cases of equation (1) where neither x nor $x+1$ is a perfect nth power. As noted previously, equations with $|A - B| = 1$ have, via Lemma 4.4, only solutions (a, b) with $|ab| = 1$. Otherwise, for each n, we consider primes of the shape $p = 2nk + 1$, noting that there are at most $(2k+1)^2$ values for $Aa^n - Bb^n$ modulo p. If none of these are 1 modulo p, we deduce a contradiction, whereby the equation $Aa^n - Bb^n = 1$ has no integral solutions. If we are unable to find a suitable p, as a last resort, we check for insolubility modulo n^2. By way of example, suppose that

$$2a^7 - 5b^7 = 1 \text{ for } a, b \in \mathbb{Z}.$$

Since $x^7 \equiv 0, \pm 1, \pm 12 \pmod{29}$, it follows that

$$2a^7 - 5b^7 \equiv 0, \pm 2, \pm 3, \pm 4, \pm 5, \pm 7, \pm 10 \pmod{29},$$

a contradiction. We refer to this approach as the *local method*. Such a technique was applied to large degree binomial Thue equations in [3], cf. also [28] and [13]. In practice, this serves to deduce insolubility for, typically, all but small values of n. For the pairs (A, B) we are left to treat, with AB as in (8), we are unable to solve only the equations corresponding to the following values of (A, B, n) :

$$\begin{gathered}(2,7,7),(2,10125,7),(3,10,7),(3,14,7),\\(6,11,7),(27,1250,7),(33,31250,7),(125,162,7),(6,11\cdot 5^{10},11),\\(3,14,13),(33,2\cdot 5^{18},19),(2\cdot 3^9,5^{10},19).\end{gathered}$$

We resolve the corresponding equations $Aa^n - Bb^n = 1$ via, for example, MAGMA. We find, for each triple, that there are no solutions in nonzero integers (a, b).

It remains, then, to treat values of (A, B) with $A = 1$; i.e. equation (1) where one of x or $x + 1$ is itself a perfect nth power.

6. Equation (1) with either x or $x + 1$ a perfect nth power

We will present three methods for resolving the remaining cases of equation (1). The second may be viewed as a computationally efficient variant of the first. Both the first and second methods apply only for certain values of k (including, for instance, k even). Our third method is applicable for every k (as it enables one to solve equations of the shape $x^n - Dy^n = 1$, in generality). In all cases, we will assume that n is bounded, say $n \leq n_0$ (which we may do, for general k, by using the estimate $n \leq ck^2 \cdot \log(2k)$ from [25] with an explicitly given c or arguing as in the proof of Proposition 4.3) and that $x, x + 1 \in \{a^n, Db^n\}$ for $D = AB$ as in (8).

Our first technique applies if either $2x + 1$ or $x^2 + x - 1$ is a factor of the polynomial $S_k(x)$ (i.e. if k is even or $k = 9$) and relies entirely upon local arguments à la Section 5. If $2x + 1$ divides $S_k(x)$ then, writing $2x + 1 = Cc^n$ for integers c and C, we have both

$$Cc^n - 2a^n = \pm 1 \quad \text{and} \quad Cc^n - 2Db^n = \pm 1. \tag{18}$$

In view of Lemma 4.4, the case $C = 1$ can be excluded. In either case, we may proceed via the methods of Section 5. Similarly, if $x^2 + x - 1$ divides $S_k(x)$, then we have $x(x + 1) = D(ab)^n$ and so

$$x^2 + x - 1 = D(ab)^n - 1.$$

Writing $x^2 + x - 1 = Ss^n$ with $s, S \in \mathbb{N}$, we obtain, if $S = 1$, two solutions in positive integers to the equation $|u^n - Dv^n| = 1$ (contradicting Lemma 4.4). It follows that $S > 1$ and hence we deduce the equation

$$D(ab)^n - Ss^n = 1, \quad \text{with} \quad \min\{D, S\} > 1.$$

This may, in practice, again be treated by local methods.

Our second method, while a refinement of the first, again, for $k \leq 10$, appears to be restricted to even values of k. We will use the following immediate consequence of Theorems 1 and 2 of Győry [11] :

Lemma 6.1. *Let $n > 3$ be a prime, and D a positive integer such that*

$$\gcd(\phi(D), n) = 1 \qquad \text{and} \qquad D^{n-1} \not\equiv 2^{n-1} \pmod{n^2}. \tag{19}$$

If x, y and z are coprime, non-zero integers such that

$$x^n + y^n = Dz^n \tag{20}$$

with n coprime to z, then

$$r^{n-1} \equiv 1 \pmod{n^2}$$

for each divisor r, of D if $n|xy$ or of Dxy otherwise.

This was proved by means of Eisenstein's reciprocity theorem in cyclomotic fields. A corollary of Lemma 6.1 is

Corollary 6.2. *Let $n > 3$ be prime and D a positive integer satisfying* (19). *If x, y and z are coprime positive integers satisfying (20), then either (i) $n|z$; or (ii) $n|xy$ and Dz is odd.*

Proof. Suppose that n is coprime to z. We may further assume that n is coprime to D. Indeed, if $n|D$ then (20) implies that $n|x + y$, whence $n|\frac{x^n+y^n}{x+y}$. Thus $n^2|D$, which contradicts (19). Further, replacing D by $D \cdot 2^{\alpha n}$, for suitable α, we may assume that z is odd. From equation (20), Dxy is even and, apart possibly from the case where D is odd and $n|xy$, Lemma 6.1 gives

$$D^{n-1} \equiv 1 \pmod{n^2} \quad \text{and} \quad 2^{n-1} \equiv 1 \pmod{n^2},$$

contrary to our assumptions. □

We note that, in the case where n is coprime to xyz, a less precise version of Lemma 6.1 can be found in a recent paper of Halberstadt and Kraus (cf. [13], Theorem 6.1).

We will apply Corollary 6.2 to the equation $a^n - Db^n = 1$, with D even and $n \geq 7$ prime, to conclude that, if n is coprime to $\phi(D)$ and

$$(D/2)^{n-1} \not\equiv 1 \pmod{n^2}, \tag{21}$$

then we necessarily have $b \equiv 0 \pmod{n}$. Since we are assuming $x, x+1 \in \{a^n, Db^n\}$, it follows that $x(x+1) \equiv 0 \pmod{n^n}$. Further,

$$T_k(x) \equiv T_k(0) \pmod{x(x+1)}.$$

If we write $T_k(x) = Tt^n$ and $2x+1 = Ss^n$ for s, t, S and T positive integers, then it follows that

$$T^{n-1} \equiv T_k(0)^{n-1} \pmod{n^2} \tag{22}$$

and

$$S^{n-1} \equiv 1 \pmod{n^2}. \tag{23}$$

If either $T \not\equiv T_k(0) \pmod{n^2}$ or $S \not\equiv 1 \pmod{n^2}$, then, as will be seen below, the congruences (22) and (23) are unlikely to hold, for prime n with $7 \leq n \leq n_0$. To deal with the remaining values of n satisfying both (22) and (23), we may turn to our methods of Section 5 or employ classical work of Maillet (see e.g. Dickson's book [9], Vol II, p. 759, item 167), at least, for the latter, provided the prime n is regular.

We will illustrate this technique applying it to the remaining cases of (1) with $k \leq 10$. In practice, if $k > 10$, we can guarantee that one of $T \not\equiv T_k(0) \pmod{n^2}$ or $S \not\equiv 1 \pmod{n^2}$, only for even values of k.

From the considerations of Section 4, we may suppose that $k \in \{4, 6, 8, 9, 10, 11\}$. We recall that $n \geq 7$. First consider the case $k \in \{4, 6, 8\}$. If $\gcd(2x+1, T_k(x))$ divides 3, then either

$$(2x+1)(2x+2) = 2^\alpha 3^\beta y_2^n \ , \text{ or } \quad 2x(2x+1) = 2^\alpha 3^\beta y_2^n,$$

for nonnegative integers α, β and a positive integer y_2. Applying Lemma 4.1, as previously, leads, in each case, to the trivial solution. Hence $\gcd(2x+1, T_k(x))$ does not divide 3. We recall that by assumption $\max\{\delta_5, \delta_7\} > 0$. It follows that we may assume

$$2x+1 = 3^{1-\delta_3} \cdot 7^{n-1} s^n, \ \ T_k(x) = 7t^n, \quad \text{if } \ k = 4, \tag{24}$$

$$2x+1 = 3^{1-\delta_3} \cdot 31^{n-1} s^n, \ \ T_k(x) = 31t^n, \quad \text{if } \ k = 6 \tag{25}$$

and

$$2x+1 = 3^{1-\delta_3} \cdot 127^{n-1} \cdot s^n, \ \ T_k(x) = 381 \cdot t^n, \quad \text{if } \ k = 8, \tag{26}$$

where $\delta_3 \in \{0, 1\}$. We deduce from (6), (8) and from $x, x+1 \in \{a^n, Db^n\}$ with $D = AB$ that $D \in \{10, 30\}$ if $k = 4$ or 8, and $D \in \{14, 42\}$ if $k = 6$. We employ now Corollary 6.2, in conjunction with the following result:

Lemma 6.3. *If u and v are positive integers with (u, v) contained in the set*

$$\{(3,1),(5,1),(5,3),(7,1),(7,3),(11,1),(15,1),(21,1),\\(31,1),(31,3),(33,1),(127,1),(127,3)\},$$

then the only solutions to the congruence

$$u^{n-1} \equiv v^{n-1} \pmod{n^2}$$

in prime n with $7 \leq n < 4000$ are given by triples (u, v, n) as follows :

$$\{(5,3,7),(5,3,383),(11,1,71),(31,1,7),(31,1,79),(31,3,29),\\(31,3,191),(31,3,431),(33,1,233),(127,1,19),(127,1,907)\}.$$

Proof. This is a routine computation which takes a matter of seconds using MAPLE 7 on a Sun Ultra. For similar numerical results, we refer to the book of Ribenboim [27]. □

We conclude that (21) holds for $D \in \{10, 14, 30, 42\}$ and $n \geq 7$ prime. Since $T_k(0) = -1, 1$ or -3, according as $k = 4, 6$ or 8, respectively, equations (24), (25), and (26), together with (22) and (23) give

$$u^{n-1} \equiv \left(3^{1-\delta_3}\right)^{n-1} \equiv 1 \pmod{n^2},$$

where $u = 7, 31$ or 127, according as $k = 4, 6$ or 8, respectively. In view of Lemma 6.3 this is possible only if $\delta_3 = 1$ and

$$(k, n) \in \{(6,7),(6,79),(8,19),(8,907)\}.$$

For these, except for $(k, n) = (6, 7)$, at least one of the corresponding equations in (18) is insoluble, modulo $4 \cdot 79 + 1, 22 \cdot 19 + 1$ or $6 \cdot 907 + 1$, respectively, which proves our Theorem 1.1 for $k = 4, 8$. If $(k, n) = (6, 7)$, then in view of $\delta_3 = 1$ we have $D = 42$, and MAGMA assures us that the Diophantine equation $a^7 - 42b^7 = 1$ has no solutions in nonzero integers, completing the proof, in case $k = 6$.

It remains to consider $k \geq 9$. If $k \in \{9, 10\}$, then $x(x+1) = Da^n$ as in (6) with $\max\{\delta_5, \delta_{11}\} > 0$. Further, $x^2 + x - 1$ divides $S_k(x)$, and it is easy to show that

$$\gcd\left(x^2 + x - 1, C_k S_k(x)/(x^2 + x - 1)\right)$$

is equal to 1 if $k = 9$, and is one of $1, 5$ or 11 if $k = 10$. It follows that $x^2 + x - 1 = Cc^n$ with $c \in \mathbb{N}$, where $C = 1$ if $k = 9$, or $C \in \{1, 5, 11\}$ if $k = 10$. Arguing as in the paragraphs preceding Lemma 6.1, if $C = 1$, we obtain a contradiction via Lemma 4.4. We thus have $k = 10$ and $C \in \{5, 11\}$. If $C = 5$, since $\max\{\delta_5, \delta_{11}\} > 0$, it follows that

$$x, x+1 \in \{a^n, 2 \cdot 3^{\delta_3} \cdot 11 \cdot b^n\},$$

i. e. $D = 22$ if $\delta_3 = 0$, or $D = 66$ if $\delta_3 = 1$. By applying Corollary 6.2 and Lemma 6.3 to $a^n \pm 1 = Db^n$, we infer that either $n = 71$ (if $\delta_3 = 0$), $n = 233$ (if $\delta_3 = 1$), or $n^2 | x(x+1)$. In this last situation,

$$5c^n = x^2 + x - 1 \equiv -1 \pmod{n^2}$$

and so $5^{n-1} \equiv 1 \pmod{n^2}$, contradicting Lemma 6.3. Considering the equations $22(ab)^{71} - 5c^{71} = 1$ and $66(ab)^{233} - 5c^{233} = 1$, modulo 569 and 467, respectively, contradicts $a, b, c \in \mathbb{Z}$, completing the proof of Theorem 1.1 in case $k = 10$, $C = 5$.

If $k = 10$ and $C = 11$, then

$$x, x+1 \in \{a^n, 2 \cdot 3^{\delta_3} \cdot 5^{n-1} \cdot b^n\}.$$

From Corollary 6.2 we deduce that either $\delta_3 = 1$ and

$$5^{n-1} \equiv 3^{n-1} \pmod{n^2}$$

so that, by Lemma 6.3, $n \in \{7, 383\}$, or $n^2 | x(x+1)$. In the latter case $x^2 + x - 1 = 11c^n$ implies that $11^{n-1} \equiv 1 \pmod{n^2}$ and so, again from Lemma 6.3, $n = 71$. In this case, if $\delta_3 = 0$,

$$(2x+1)(3x^6 + 9x^5 + 2x^4 - 11x^3 + 3x^2 + 10x - 5) = 15d^{71}$$

for some integer d and so, since

$$3x^6 + 9x^5 + 2x^4 - 11x^3 + 3x^2 + 10x - 5 \equiv -5 \pmod{x(x+1)},$$

we have $15d^{71} \equiv \pm 5 \pmod{71^2}$, whence $3^{70} \equiv 1 \pmod{71^2}$, a contradiction. It remains to deal with the equations

$$2 \cdot 3 \cdot 5^{n-1} \cdot (ab)^n - 11c^n = 1, \;\; n \in \{7, 71, 383\}.$$

Since these have no solutions modulo 29 if $n = 7$, modulo 569 if $n = 71$ and modulo 4597 if $n = 383$, this completes the proof of Theorem 1.1 in case $k = 10$.

If $k = 11$, our prior work implies that

$$x, x+1 \in \{a^n, 2 \cdot 3^{\delta_3} \cdot 5^{(n-1)/2} \cdot b^n\}, \;\; \delta_3 \in \{0, (n+1)/2\}$$

and so, if $\delta_3 = 0$, $T_{11}(x) = 30t^n$ for some positive integer t. Since

$$\left(5^{(n-1)/2}\right)^{n-1} \equiv 1 \pmod{n^2}$$

implies that $5^{n-1} \equiv 1 \pmod{n^2}$, Corollary 6.2 and Lemma 6.3 yield $n^2 | x(x+1)$. So, from (22), $3^{n-1} \equiv 1 \pmod{n^2}$, contradicting Lemma 6.3. We thus have that $\delta_3 = (n+1)/2$. In this case, the assumption that $x(x+1) \equiv 0 \pmod{n^2}$ fails to lead to a contradiction, as (22) is satisfied. In the next section, we will show how a new method for solving high degree binomial Thue equations, based upon the theory of Frey curves and modular forms, can be applied to finish the proof of Theorem 1.1, in case $k = 11$, by solving (unconditionally) the equation

$$a^n - 2 \cdot 3^{(n+1)/2} \cdot 5^{(n-1)/2} \cdot b^n = 1. \tag{27}$$

7. Frey curves

To treat a Diophantine equation of the shape $a^n - Db^n = 1$, it may prove profitable to view this as a specialization of a ternary equation of the form $a^n - Db^n = c^m$. For $m \in \{2, 3, n\}$, there exist established techniques for treating such equations, via Frey curves and the theory of Galois representations and modular forms, described in detail in [4], [5], [13] and [15]. As the example (27) we are choosing leads to a curve with a high power of 3 in its discriminant, for technical reasons, we will adopt the approach of [5]. The main features of the general method will be apparent in this case.

Let us suppose we have a solution in nonzero integers a and b to equation (27) (so that $|ab| > 1$), with $n \geq 7$ prime, and consider the elliptic curve

$$E: \quad y^2 + 3xy - Db^n y = x^3,$$

where $D = 2 \cdot 3^{(n+1)/2} \cdot 5^{(n-1)/2}$. From Lemma 3.4 of [5], the Galois representation

$$\rho_{E,n} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_n),$$

on the n-torsion points $E[n]$ of the elliptic curve E, arises from a cuspidal newform f of weight 2, trivial character and level 30. More precisely, for our purposes, if $p > 5$ is a prime dividing b, we have that

$$\mathrm{trace}\rho_{E,n}\,(\mathrm{Frob}_p) = p + 1$$

and hence, if we denote the pth Fourier coefficient of an elliptic curve of conductor 30 by c_p, it follows that

$$c_p \equiv p + 1 \pmod{n}. \tag{28}$$

As in Section 5, we search for a local obstruction, by considering (27) modulo a prime of the form $p = 2kn+1$, coprime to D, for $k \in \mathbb{N}$, under the additional assumption that p fails to divide b. For such a prime, there are now at most $4k^2 + 2k$ possible residue classes for $a^n - Db^n$ with p coprime to b. If none of these are congruent to 1 modulo p, we may conclude that p divides b. In this case, since $p = 2kn + 1$, (28) implies that

$$c_p \equiv p + 1 \equiv 2 \pmod{n}. \tag{29}$$

where, again, c_p is the pth Fourier coefficient of an elliptic curve of conductor 30 (denoted $30A$ in Cremona's tables [7]). If this fails to occur, then we reach our desired conclusion. In particular, if k is not too large, relative to n, say $k \leq \frac{n-5}{8}$, then the Hasse-Weil bounds imply that $|c_p| < n-2$ and so we obtain a contradiction, unless $c_p = 2$.

We carry out this procedure for all primes $7 \leq n < 4000$. The computation took under a minute on a Sun Ultra 10; full data is available from the authors on request. We are able to find a prime $p = 2nk+1$ from which we can conclude that (27) has no solutions in positive integers (a,b) with b coprime to p. In all cases, there is such a prime with $k \leq \frac{n-5}{8}$, except for those (a,b) and n listed below, where we have also tabulated the smallest viable p and corresponding c_p:

n	p	c_p	n	p	c_p
7	29	-6	17	103	-4
11	23	0	19	191	-24
13	53	-6	31	311	0

Note that in each of these cases, c_p fails to satisfy (29). It follows that we may assume that $c_p = 2$. Since, for all primes n with $7 \leq n < 4000$, we can find a p with the desired properties and $23 \leq p \leq 229981$ (where this last value corresponds to $n = 3833$), it suffices to consider those primes p for which $c_p = 2$, in this range. There are precisely 127 such primes (ranging from $p = 37$ to $p = 229681$). The only one of these that provides a minimal local obstruction in the above sense is $p = 29077$ (which plays such a role if $n = 2423$). Considering equation (27) with $n = 2423$ modulo 33923, leads, however, to the conclusion that $b \equiv 0 \pmod{33923}$, contradicting the fact that $c_{33923} = -180$. This completes the proof of Theorem 1.1.

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